Probing Nuclear Superfluidity and Superconductivity with Neutron Stars

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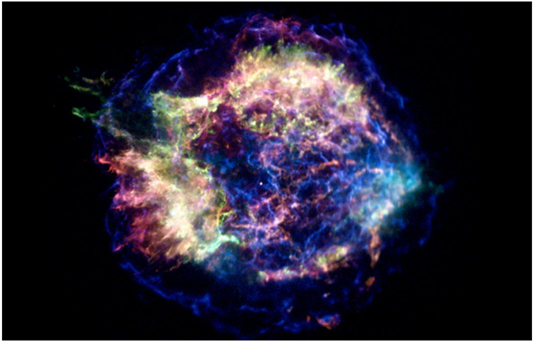
Neutron stars: laboratories for dense matter

Formed in gravitational core-collapse supernova explosions, neutron stars are the **most compact stars** in the Universe.



Nuclear physics:

$M \sim 1 - 2M_\odot$
$R \sim 10$ km
$\Rightarrow \rho \sim 10^{15}$ g cm^{-3}

Energy scale: MeV

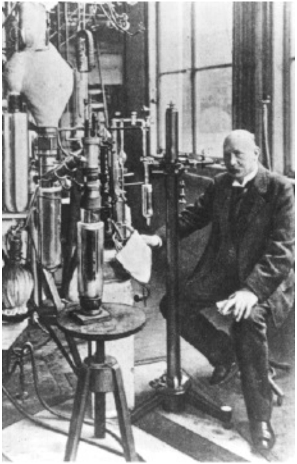
"cold" $\lesssim 10^{10}$ K $\lesssim$ "hot"

Neutron stars are initially very hot ($\sim 10^{12}$ K) but cool down to $\sim 10^9$ K within days by releasing neutrinos.

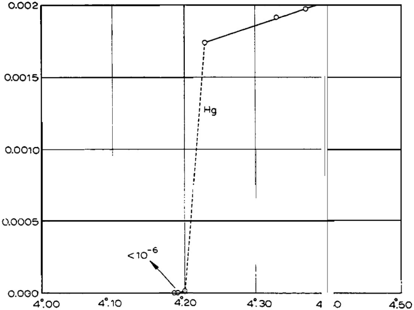
Their dense matter is thus expected to undergo various phase transitions, as observed in terrestrial materials at low-temperatures.

Discovery of "suprageleider"

Heike Kamerlingh Onnes and his collaborators were the first to liquefy helium ($T = 4.2$ K) in 1908.



On April 8th, 1911, H. K. Onnes and Gilles Holst discovered that the **electric resistance of mercury vanished** at $T_c \simeq 4.2$ K

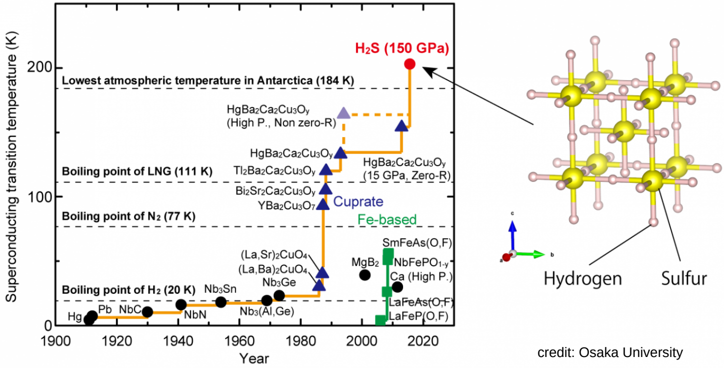


Onnes was awarded the Nobel Prize in 1913.

The year later, tin and lead were found to be also superconducting.

Timeline of superconductor discoveries

High-T_c cuprate superconductors were discovered in 1986 by IBM researchers G. Bednorz and K.A. Müller (Nobel Prize in 1987).



credit: Osaka University

Recently CH_8S has been found to be superconducting (under “high” pressures) at room temperatures

Snider et al., Nature 586, 373 (2020)

Discovery of superfluidity

During the 1930s, several groups found that below $T_\lambda = 2.17$ K, helium (He II) does not behave like an ordinary liquid: can flow without resistance, does not boil, flow from cool to hot regions.

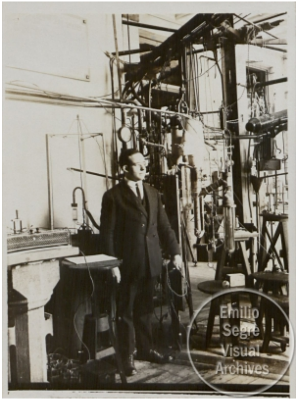
"**by analogy with superconductors**, *the helium below the* λ*-point enters a special state which might be called* ***superfluid**.*"
Kapitza, Nature 141, 74 (1938)

Kapitza received the Nobel Prize in 1978.

"*the observed type of flow most certainly* **cannot be treated as laminar or even as ordinary turbulent flow**."
Allen & Misener, Nature 141, 75 (1938)

Following London's suggestion that He II is related to BEC, Tisza proposed the two-fluid model later developed by Landau.

Onnes and Dana observations about liquid helium



Onnes and his collaborators discovered superfluidity without realizing it the same day they discovered superconductivity in 1911!

Onnes noted about liquid helium:
"*Just before the lowest temperature [about 1.8 K] was reached, the boiling suddenly stops...*"

Leo Dana, a visiting student at Onnes' lab measured the lambda transition in the specific heat in 1922 but no one paid attention!

About the history of superfluidity and superconductivity:

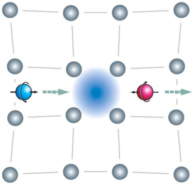
Balibar in "History of Artificial Cold, Scientific, Technological and Cultural Issues", Boston Studies in the Philosophy and History of Science 299 (Springer, 2014)

van Delft&Kes, Phys. Today 63, 9, 38 (2010)

Balibar, J. Low Temp. Phys. 146, 441 (2007)

Bardeen-Cooper-Schrieffer theory

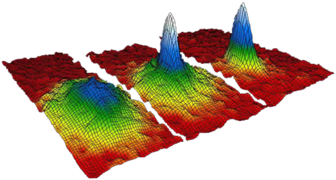
The microscopic theory of Bardeen, Cooper and Schrieffer was published in 1957 (1972 Nobel Prize in Physics)



The dynamical distorsions of the crystal lattice (phonons) can induce an **attractive effective interaction** between electrons of opposite spins.

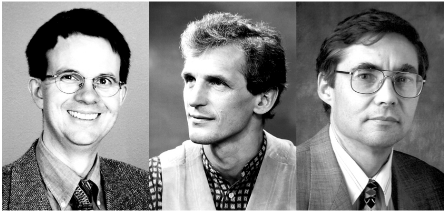
- Electrons form **bosonic pairs** and can thus **condense** below T_c. A superconductor can thus be viewed as a charged superfluid.
- This suggested that fermionic atoms could be superfluid. In 1971, Osheroff discovered ^{3}He superfluidity below 2.5 mK.

Bose-Einstein condensation in dilute atomic gases



On June 5, 1995, the first BEC was produced by Eric Cornell and Carl Wieman at the University of Colorado at Boulder NIST-JILA, with $\sim$ 2000 rubidium ^{87}Ru atoms cooled to 170 nK.

Shortly thereafter, Wolfgang Ketterle's team at MIT obtained a BEC of $\sim 5 \times 10^5$ sodium ^{23}Na atoms cooled to 2 μK.



For their achievements, Cornell, Ketterle and Wieman were awarded the 2001 Nobel Prize in Physics.

On December 16, 2003, the first fermionic condensate was produced by Deborah Jin at JILA with 5×10^5 potassium ^{40}K atoms at 50 nK.

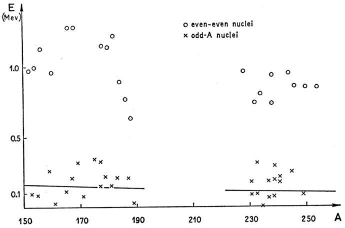
Nuclear superfluidity and superconductivity

The implications of the BCS theory (published in January 1957) for atomic nuclei were first discussed by A. Bohr, B. R. Mottelson, and D. Pines during the Summer of 1957.

D. Pines in "BCS: 50 Years" (World Scientific, 2011), pp.85-105

Bohr, Mottelson, and Pines speculated that nuclear pairing might explain the **energy gap** in the excitation spectra of nuclei.

Phys. Rev. 110, 936 (1958)

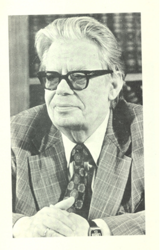


They also anticipated that nuclear pairing could explain **odd-even mass staggering, and the reduced moments of inertia** of nuclei.

"**The present data are insufficient to indicate the limiting value for the gap in a hypothetical infinitely large nucleus.**"
Bohr, Mottelson, Pines.

Superfluidity and superconductivity in neutron stars

In the 1960's, several superconductors were known but only one superfluid, ^{4}He (superfluidity of ^{3}He was discovered at the beginning of the 1970's).



Bogoliubov, who developed a microscopic theory of superfluidity and superconductivity, was the first to explore its application to nuclear matter.

Dokl. Ak. nauk SSSR 119, 52 (1958)

Neutron-star superfluidity was predicted by Arkady Migdal in 1959, and first studied by Ginzburg & Kirzhnits in 1964 **before the discovery of pulsars** in 1967.

Migdal, Nucl. Phys. 13, 655 (1959)

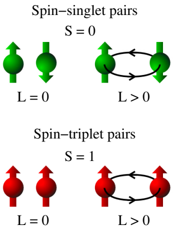
Ginzburg & Kirzhnits, Zh. Eksp. Teor. Fiz. 47, 2006 (1964)

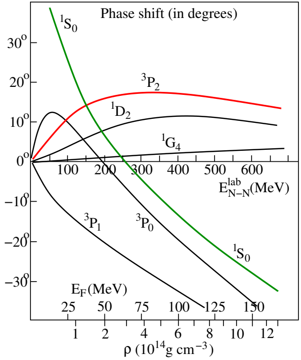
Neutron pairing channels

Below $\sim 10^{10}$ K, neutrons form pairs and condense into a superfluid phase

Pairing ($^{2S+1}L_J$) if $\delta > 0$

Gezerlis, Pethick, Schwenk in Novel Superfluids, Vol.2, Bennemann&Ketterson (Oxford University Press, 2014), Chap. 22





Microscopic calculations:

- diagrammatic methods
- variational methods
- quantum Monte Carlo methods.

For a recent review: *Sedrakian & Clark, Eur. Phys. J. A55, 167 (2019)*

$^1S_0$ pairing in neutron matter: BCS and beyond

Different many-body methods lead to comparable predictions:

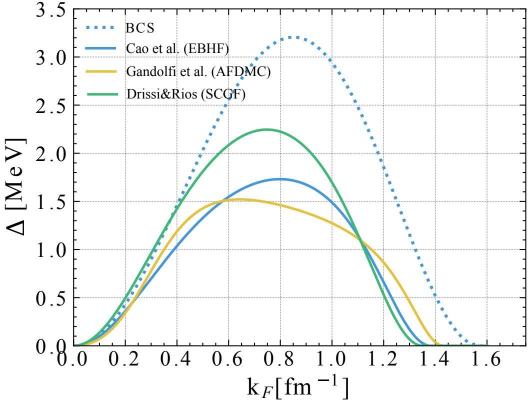
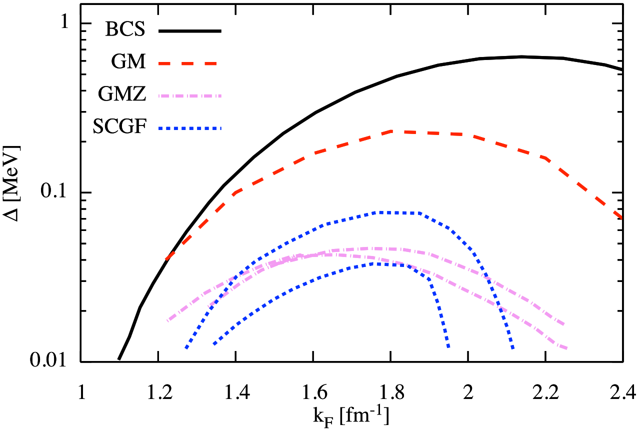


figure made by V. Allard

The $^1S_0$ pairing gaps are reduced by medium effects.

$^3PF_2$ pairing in neutron matter

The $^3PF_2$ pairing gaps are also suppressed by medium effects, but are more uncertain:



Sedrakian & Clark, Eur. Phys. J. A55, 167 (2019)

Superfluid and superconducting phases

The interior of a neutron star is not only made of neutrons, but consists of protons, leptons, hyperons, and possibly mesons, and even deconfined quarks!

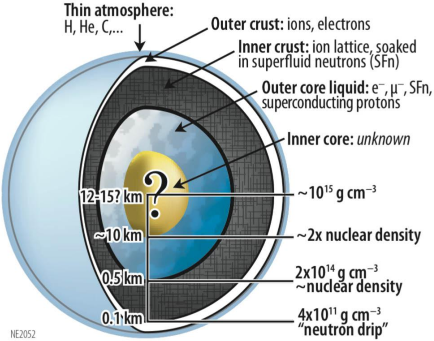
Possible phases:

- $^1S_0$ and $^3PF_2$ proton pairing
- neutron-proton pairing
- hyperon-hyperon pairing ($^1S_0$ $\Lambda\Lambda$)
- hyperon-nucleon pairing ($^1S_0$ $n\Lambda$, $^1S_0$ $n\Sigma^-$, $^3SD_1$ $n\Sigma^-$)
- quark pairing

Although $^1S_0$ proton superconductivity is well established, the other superfluid/superconducting phases are more uncertain.

Superstars

The huge gravity of neutron stars produces the highest-T_c and largest superfluids and superconductors known in the Universe!



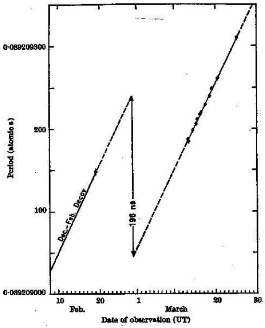
Neutron stars	$\sim 10^{10}$ **K**
$\vdots$	$\vdots$
CH_8S	288 K
Cuprates	1 – 130 K
Electrons in ordinary metals	1 – 25 K
Helium-4	2.17 K
Helium-3	2.491×10^{-3} K
Bosonic condensates	$\sim 10^{-6}$ K
Fermionic condensates	$\sim 10^{-8}$ K

Predicted long ago, these quantum phases may be probed through astrophysical observations.

Pulsar frequency glitches and superfluidity

Pulsars are spinning very rapidly with **extremely stable periods** $\dot{P} \gtrsim 10^{-21}$, outperforming the best atomic clocks.

Milner et al., Phys. Rev. Lett. 123, 173201 (2019)



Still, some pulsars have been found to **suddenly spin up** (in less than a minute).

651 glitches have been detected in 207 pulsars.

`http://www.jb.man.ac.uk/pulsar/glitches.html`

Recent review: *Manchester, Proc. IAU 13 (2017)*

The first glitch was detected in Vela in 1969.
Radhakrishnan&Manchester, Nature 222, 228 (1969)
Reichley&Downs, ibid. 229

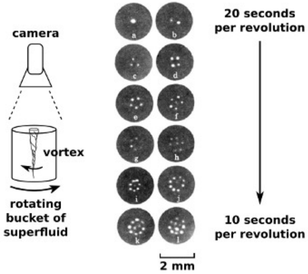
The very **long spin-down relaxation** (up to years) provided the first evidence for superfluidity.

Baym, Pethick, Pines, Nature 224, 673 (1969)

Vortex dynamics in neutron stars

A rotating superfluid is threaded by **quantized vortex lines**, each of which carries an angular momentum $\hbar$.

Similarly, a rotating neutron star is threaded by $\sim 10^{18}$ vortices, as pointed out by Ginzburg & Kirzhnits in 1964.



Yarmchuk et al., PRL43, 214 (1979)

In 1975, it was proposed that giant glitches are triggered by the sudden **unpinning of vortices** in neutron-star crust.
Anderson&Itoh, Nature 256, 25 (1975)

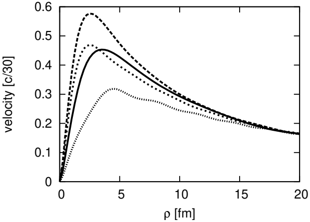
This scenario found support from **laboratory experiments** on He II.
J. S. Tsakadze & S. J. Tsakadze, J. Low Temp. Phys. 39, 649 (1980)

Postglitch relaxation can be explained by **vortex creep**.
Pines & Alpar, Nature 316, 27(1985)

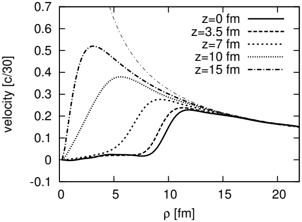
Vortex pinning

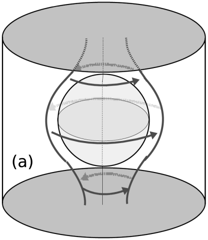
Neutron superfluid vortices can pin to nuclei in the crust:

single vortex



single vortex + cluster





Avogadro et al, Nucl.Phys.A811,378(2008)

Microscopic calculations of pinning forces:

- local density approximation and semi-classical methods
- nuclear energy density functional theory.

Pinning depends on the structure of the crust, on the rigidity of the lines and on the vortex dynamics.
Wlazlowski et al., PRL 117, 232701 (2016)

Glitches and the superfluid inertia

Giant glitches are thus interpreted as **sudden transfers of angular momentum between the superfluid and the rest of star**.

The fractional moment of inertia of the superfluid component can be inferred from **pulsar-timing observations**:

$$\frac{I_s}{I} \geq \mathcal{G} \, , \qquad \mathcal{G} = 2\tau_c A_g$$

$\tau_c = \dfrac{\Omega}{2|\dot{\Omega}|}$ is the characteristic age,

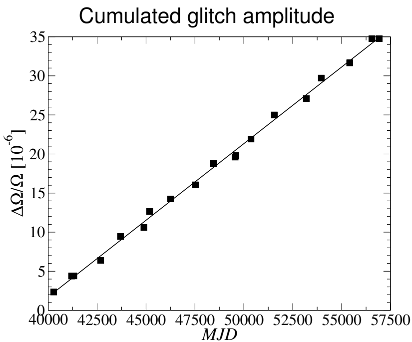
$A_g = \dfrac{1}{t}\displaystyle\sum_i \dfrac{\Delta\Omega_i}{\Omega}$ is the glitch activity.

Link, Epstein, Lattimer, Phys. Rev. Lett. 83, 3362 (1999)

Further information can be gained from individual glitches but more model dependent.

Pulsar glitch constraint

Since 1969, 22 glitches have been regularly detected in Vela. The latest one occurred in July 2021.



The cumulated glitch amplitude increases almost linearly:

$$\sum_i \frac{\Delta\Omega_i}{\Omega} = A_g t$$

where $A_g \simeq 2.25 \times 10^{-14}\ \mathrm{s}^{-1}$

$$\Rightarrow \mathcal{G} = 2\tau_c A_g \simeq 1.62\%$$

The analysis of other glitching pulsars leads to $\mathcal{G} \lesssim 2\%$.

Neutron-star cores are expected to be superfluid. Why is $\mathcal{G}$ so small?

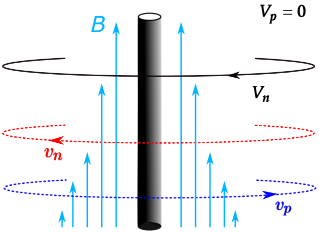
Entrainment and dissipation in neutron-star cores

Neutrons and protons are **mutually entrained** similarly to superfluid ^{3}He-^{4}He mixture (Andreev-Bashkin effects).

Andreev & Bashkin, Sov. Phys. JETP 42, 164 (1975)

Mass currents $\boldsymbol{\rho_q}$ are not aligned with superfluid velocities $\boldsymbol{V_q}$:

$$\boldsymbol{\rho_n} = \rho_{nn}\boldsymbol{V_n} + \rho_{np}\boldsymbol{V_p}$$
$$\boldsymbol{\rho_p} = \rho_{pn}\boldsymbol{V_n} + \rho_{pp}\boldsymbol{V_p}$$



Neutron vortices thus carry a **fractional magnetic quantum flux**

$$\Phi_\star = \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{\ell} = k\Phi_0\,, \quad k = \frac{\rho_{pn}}{\rho_{pp}}, \Phi_0 \equiv \frac{hc}{2e}$$

Sedrakyan&Shakhabasyan, Astrofizika 8, 557 (1972); ibid. 16, 727 (1980)

Due to electrons scattering off the magnetic field of the vortex lines, **the core superfluid is strongly coupled to the crust**.

Alpar, Langer, Sauls, ApJ 282, 533 (1984)

Time-dependent Hartree-Fock-Bogoliubov theory

The dynamics of the neutron-proton superfluid mixture is described by the **time-dependent Hartree-Fock-Bogoliubov equations**:

$$\begin{pmatrix} h_q(\boldsymbol{r},t)-\lambda_q & \Delta_q(\boldsymbol{r},t) \\ \Delta_q(\boldsymbol{r},t)^* & -h_q(\boldsymbol{r},t)^*+\lambda_q \end{pmatrix} \begin{pmatrix} \psi_1^{(q)}(\boldsymbol{r},t) \\ \psi_2^{(q)}(\boldsymbol{r},t) \end{pmatrix} = \mathrm{i}\hbar\frac{\partial}{\partial t} \begin{pmatrix} \psi_1^{(q)}(\boldsymbol{r},t) \\ \psi_2^{(q)}(\boldsymbol{r},t) \end{pmatrix}$$

$$h_q(\boldsymbol{r},t) \equiv -\boldsymbol{\nabla}\cdot\frac{\hbar^2}{2m_q^{\oplus}(\boldsymbol{r},t)}\boldsymbol{\nabla} + U_q(\boldsymbol{r},t) - \frac{\mathrm{i}}{2}\left\{\boldsymbol{I_q}(\boldsymbol{r},t),\boldsymbol{\nabla}\right\} + \dots$$

$$\frac{\hbar^2}{2m_q^{\oplus}(\boldsymbol{r},t)} = \frac{\delta E}{\delta\tau_q(\boldsymbol{r},t)}, \quad U_q(\boldsymbol{r},t) = \frac{\delta E}{\delta n_q(\boldsymbol{r},t)}, \quad \boldsymbol{I_q}(\boldsymbol{r},t) = \frac{\delta E}{\delta \boldsymbol{j_q}(\boldsymbol{r},t)}$$

$$\Delta_q(\boldsymbol{r},t) = 2\frac{\delta E}{\delta\widetilde{n_q}(\boldsymbol{r},t)^*} = |\Delta_q(\boldsymbol{r},t)|e^{\mathrm{i}\phi_q(\boldsymbol{r},t)}$$

with mean fields defined via the particle and pair density matrices (thermal averages) expressible in terms of $\psi_1^{(q)}(\boldsymbol{r},t)$ and $\psi_2^{(q)}(\boldsymbol{r},t)$

$$n_q(\boldsymbol{r},\sigma;\boldsymbol{r'},\sigma';t) =< c_q(\boldsymbol{r'},\sigma';t)^\dagger c_q(\boldsymbol{r},\sigma;t) >$$

$$\widetilde{n}_q(\boldsymbol{r},\sigma;\boldsymbol{r'},\sigma';t) = -\sigma' < c_q(\boldsymbol{r'},-\sigma';t) c_q(\boldsymbol{r},\sigma;t) >$$

Superfluid velocities are not true velocities

The **superfluid velocity** defined through the phase of the pairing field

$$\Delta_q(\boldsymbol{r}, t) = |\Delta_q(\boldsymbol{r}, t)| e^{\mathrm{i}\phi_q(\boldsymbol{r}, t)}$$

by

$$\boldsymbol{V_q}(\boldsymbol{r}, t) = \frac{\hbar}{2m_q} \nabla \phi_q(\boldsymbol{r}, t)$$

is neither equal to $\hbar \boldsymbol{j_q}/\rho_q$ nor to the **true velocity**

$$\boldsymbol{v_q}(\boldsymbol{r}, t) = \frac{m}{m_q^{\oplus}(\boldsymbol{r}, t)} \frac{\hbar \boldsymbol{j_q}(\boldsymbol{r}, t)}{\rho_q(\boldsymbol{r}, t)} + \frac{\boldsymbol{I_q}(\boldsymbol{r}, t)}{\hbar}$$

associated with **mass transport**

$$\frac{\partial \rho_q}{\partial t} + \boldsymbol{\nabla} \cdot (\rho_q \boldsymbol{v_q}) = 0$$

Allard & Chamel, PRC103, 025804 (2021)

Entrainment in neutron-proton mixture

In **homogeneous matter with stationary flows**, the TDHFB equations (in the normal-fluid rest frame) are exactly solvable:

$$\boldsymbol{\rho_q} \equiv \rho_q \boldsymbol{v_q} = \sum_{q'} \rho_{qq'} \boldsymbol{V_{q'}}$$

At low temperatures and small currents, $\rho_{qq'}$ is completely determined by isovector effective mass, $\rho = \rho_n + \rho_p$, and $\eta = (\rho_n - \rho_p)/\rho$:

$$\rho_{nn} = \frac{1}{2}\rho\,(1+\eta) - \frac{1}{4}\rho\,(1-\eta^2)\left(1 - \frac{m}{m_v^\oplus}\right)$$

$$\rho_{pp} = \frac{1}{2}\rho\,(1-\eta) - \frac{1}{4}\rho\,(1-\eta^2)\left(1 - \frac{m}{m_v^\oplus}\right)$$

$$\rho_{np} = \frac{1}{4}\rho\,(1-\eta^2)\left(1 - \frac{m}{m_v^\oplus}\right) = \rho_{pn}$$

$\rho_{qq'}$ is independent of Δ_q and can be calculated within TDHF!

Chamel& Allard, PRC 100, 065801 (2019)

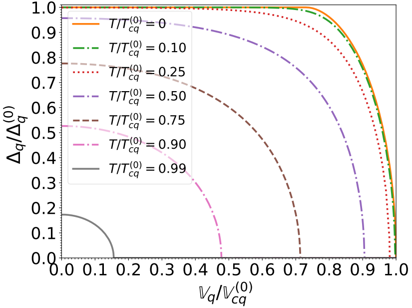
Entrainment in neutron-proton mixture

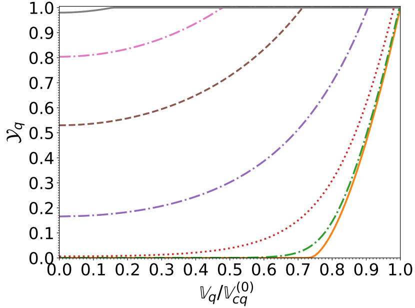
Exact solution for arbitrary temperature and currents:

Allard & Chamel, PRC103, 025804 (2021)

Universality of pairing gaps Δ_q and quasiparticle fractions $\mathcal{Y}_q$ using the **effective superfluid velocity**

$$\mathbb{V}_{\boldsymbol{q}} \equiv \frac{m}{m_q^{\oplus}} \boldsymbol{V_q} + \frac{\boldsymbol{I_q}}{\hbar}$$

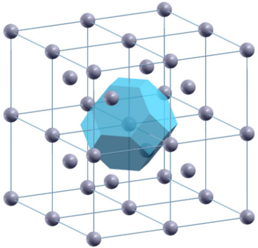




Allard & Chamel, Universe 7(12), 470 (2021)

Superfluidity in neutron-star crusts

The inner crust of neutron stars is permeated by a neutron superfluid.



Floquet-Bloch theorem:

$$\psi_{1\alpha\boldsymbol{k}}(\boldsymbol{r}+\boldsymbol{\ell}) = e^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{\ell}}\psi_{1\alpha\boldsymbol{k}}(\boldsymbol{r})$$

$$\psi_{2\alpha\boldsymbol{k}}(\boldsymbol{r}+\boldsymbol{\ell}) = e^{\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{\ell}}\psi_{2\alpha\boldsymbol{k}}(\boldsymbol{r})$$

for any lattice vector $\boldsymbol{\ell}$.

band index α: rotational symmetry
wave vector $\boldsymbol{k}$: translational symmetry.

The HFB equations need to be solved only in the **Wigner Seitz cell** with $\boldsymbol{k}$ restricted to the first Brillouin zone.

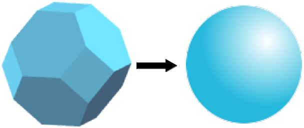
3D HFB computations remain expensive:

- Lattice spacing can be large ~ 100 fm
- Huge number of neutrons in the Wigner-Seitz cell ($\sim 10^2 - 10^3$)

HFB in the shallow layers of neutron-star crusts

At densities $\bar{n} \lesssim 0.02\ \text{fm}^{-3}$, the neutron superfluid remains dilute.

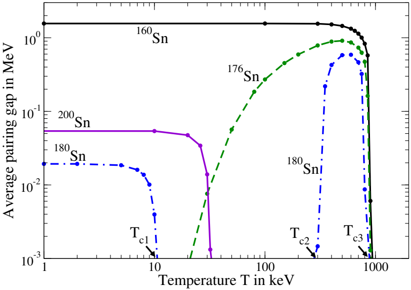
$\boldsymbol{k} = 0$ with approximate boundary conditions (1D HFB):



Sandulescu et al., PRC69, 045802 (2004)

Grill et al., PRC 84, 065801 (2011)

Pastore, PRC 91, 015809 (2015)



Margueron&Khan, PRC86, 065801(2012)

Medium effects and collective excitations:

Grasso et al.,Nucl. Phys.A807,1(2008); Baroni et al.,Phys.Rev.C82,015807(2010); Inakura&Matsuo,Phys. Rev. C99, 045801 (2019)

- This approach becomes less reliable with increasing $\bar{n}$
- and cannot describe superflow (discrete states).

From HFB to multi-band BCS theory

Due to **proximity effects**, $\Delta(\boldsymbol{r})$ varies smoothly in the densest layers: $\psi_{1\alpha\boldsymbol{k}} \approx U_{\alpha\boldsymbol{k}}\varphi_{\alpha\boldsymbol{k}}$, $\psi_{2\alpha\boldsymbol{k}} \approx V_{\alpha\boldsymbol{k}}\varphi_{\alpha\boldsymbol{k}}$, where $h(\boldsymbol{r})\varphi_{\alpha\boldsymbol{k}}(\boldsymbol{r}) = \varepsilon_{\alpha\boldsymbol{k}}\varphi_{\alpha\boldsymbol{k}}(\boldsymbol{r})$.

The HFB equations reduce to the **multiband BCS gap equations**:

$$\Delta_{\alpha\boldsymbol{k}} = -\frac{1}{2}\sum_{\beta}\int \frac{\mathrm{d}^3\boldsymbol{k'}}{(2\pi)^3}\bar{v}^{\mathrm{pair}}_{\alpha\boldsymbol{k}\alpha-\boldsymbol{k}\beta\boldsymbol{k'}\beta-\boldsymbol{k'}}\frac{\Delta_{\beta\boldsymbol{k'}}}{E_{\beta\boldsymbol{k'}}}\tanh\frac{E_{\beta\boldsymbol{k'}}}{2k_{\mathrm{B}}T}$$

$$\bar{v}^{\mathrm{pair}}_{\alpha\boldsymbol{k}\beta\boldsymbol{k'}} = \int \mathrm{d}^3r\, v^{\pi}[n_n(\boldsymbol{r}), n_p(\boldsymbol{r})]\,|\varphi_{\alpha\boldsymbol{k}}(\boldsymbol{r})|^2|\varphi_{\beta\boldsymbol{k'}}(\boldsymbol{r})|^2$$

where $E_{\alpha\boldsymbol{k}} = \sqrt{(\varepsilon_{\alpha\boldsymbol{k}}-\mu)^2+\Delta^2_{\alpha\boldsymbol{k}}}$ and $v^{\pi}[n_n(\boldsymbol{r}), n_p(\boldsymbol{r})]$ is an effective pairing interaction. The HFB solutions are

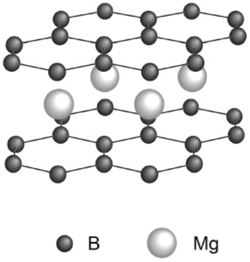
$$U_{\alpha\boldsymbol{k}} = \frac{1}{\sqrt{2}}\sqrt{1+\frac{\varepsilon_{\alpha\boldsymbol{k}}-\mu}{E_{\alpha\boldsymbol{k}}}}, \quad V_{\alpha\boldsymbol{k}} = -\frac{1}{\sqrt{2}}\sqrt{1-\frac{\varepsilon_{\alpha\boldsymbol{k}}-\mu}{E_{\alpha\boldsymbol{k}}}}$$

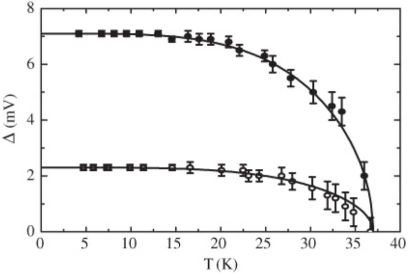
Chamel et al., Phys.Rev.C81,045804 (2010)

Multi-band BCS superconductors

Multi-band superconductors were first studied in 1959 but clear evidence were found only in 2001 with the discovery of MgB_2.

Electrons in different bands feel different pairing interactions leading to different pairing gaps:



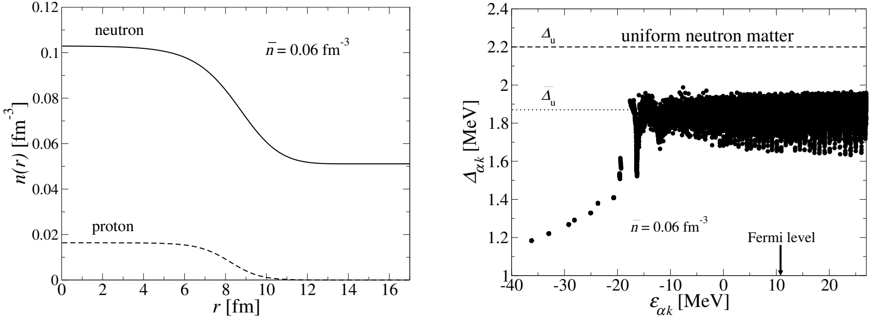


X. X. Xi, Rep. Prog. Phys.71, 116501 (2008)

In the crust of a neutron star, the number of bands involved is $\sim 10^2 - 10^3$ due to strong attractive nuclear pairing interactions!

Multi-band BCS neutron superfluid

Wigner-Seitz cell with $Z = 40$, $N = 1220$ (body-centered cubic lattice)

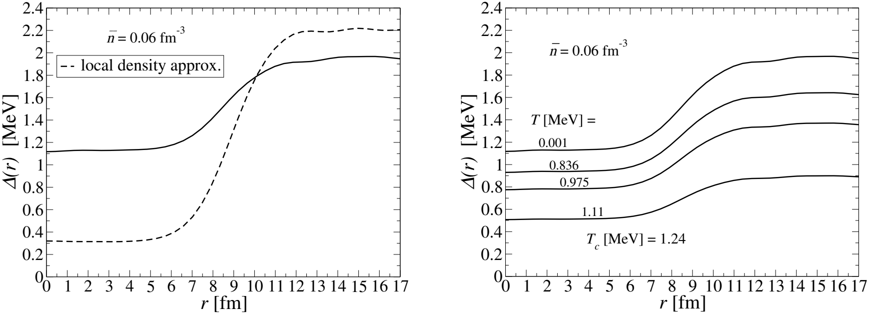


Chamel et al., Phys.Rev.C81,045804 (2010)

- Nuclear clusters lower the gap by $\sim 10 - 20\%$
- $\Delta_{\alpha \boldsymbol{k}}(T)/\Delta_{\alpha \boldsymbol{k}}(0)$ is given by the same function of T for all bands
- The critical temperature is given by the BCS relation $T_c \simeq 0.567\bar{\Delta}_u$

Multi-band BCS neutron superfluid

Wigner-Seitz cell with $Z = 40$, $N = 1220$ (body-centered cubic lattice)



Chamel et al., Phys.Rev.C81,045804 (2010)

- Both bound and unbound neutrons contribute to superfluidity
- Superfluidity permeates clusters (loosely bound Cooper pairs)
- The superfluid becomes homogeneous as T approaches T_c

Neutron superfluid fraction

Due to inhomogeneities, the neutron superfluid fraction is reduced:

$$\boldsymbol{\rho_n} = m_n n_n^s \boldsymbol{V_n} \text{ (in the crust frame).}$$

The neutron superfluid density at $T = 0$ is given by

$$n_n^s = \frac{m_n}{24\pi^3\hbar^2} \sum_\alpha \int |\boldsymbol{\nabla}_{\boldsymbol{k}} \varepsilon_{\alpha\boldsymbol{k}}|^2 \frac{\Delta_{\alpha\boldsymbol{k}}^2}{\sqrt{(\varepsilon_{\alpha\boldsymbol{k}} - \mu)^2 + \Delta_{\alpha\boldsymbol{k}}^2}^3} \mathrm{d}^3k$$

In the **weak coupling** limit $\Delta_{\alpha\boldsymbol{k}}/\mu \to 0$, it is fully determined by the shape of the Fermi surface **independently of pairing**:

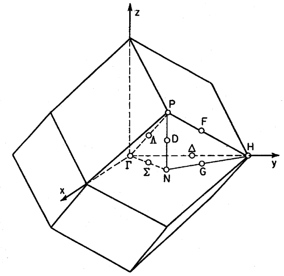
$$n_n^s \approx \frac{m_n}{24\pi^3\hbar^2} \sum_\alpha \int_{\mathrm{F}} |\boldsymbol{\nabla}_{\boldsymbol{k}} \varepsilon_{\alpha\boldsymbol{k}}| \mathrm{d}\mathcal{S}^{(\alpha)}$$

Recent review: *Chamel, J. Low Temp. Phys. 189, 328 (2017)*

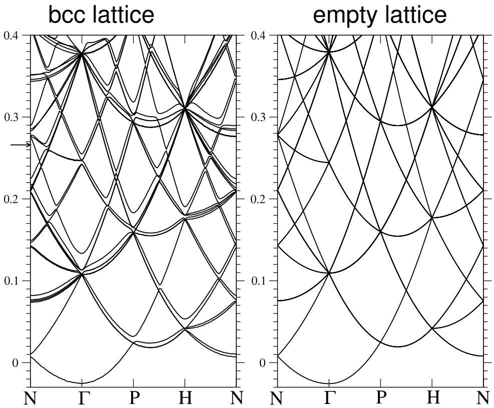
Neutron superfluid fraction in shallow region

Neutron band structure (s.p. energy in MeV vs $\boldsymbol{k}$) in a body-centered cubic (bcc) lattice at $\bar{n} = 0.0003$ fm^{-3} ($Z = 50, A = 200$):

First Brillouin zone:



Chamel,Phys.Rev.C85,035801(2012)

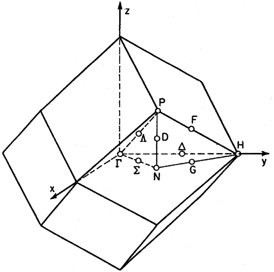


The spectrum is similar that of free neutrons: $n_n^s/n_n = 83\%$.

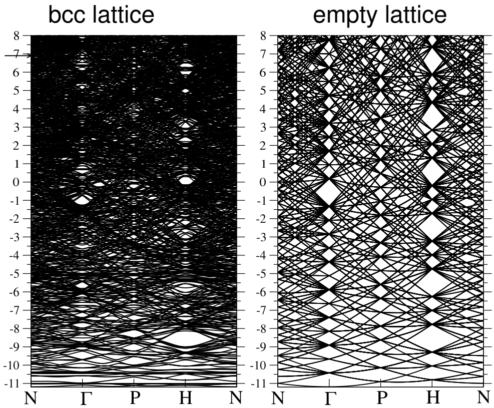
Neutron superfluid fraction in deep region

Neutron band structure (s.p. energy in MeV vs $\boldsymbol{k}$) in a body-centered cubic (bcc) lattice at $\bar{n} = 0.03$ fm^{-3} ($Z = 40, A = 1590$):

First Brillouin zone:



Chamel,Phys.Rev.C85,035801(2012)

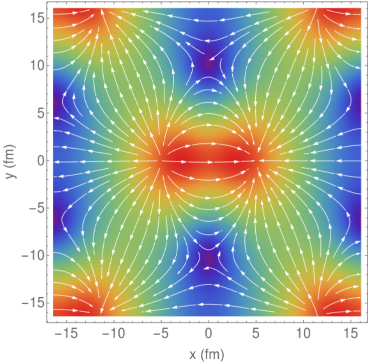


The spectrum is very different: $n_n^s/n_n = 7\%$. Neutron superfluidity is almost entirely suppressed!

Hydrodynamical approach

The neutron flow was studied in the **strong coupling limit** adopting a purely **classical hydrodynamical** treatment.

- Superfluid "velocity": $\boldsymbol{V_n} = \dfrac{\hbar}{2m_n}\nabla\Phi$
- Incompressible superfluid flow: $\nabla \cdot \boldsymbol{V_n} = 0$
- Spherical clusters (obstacles) with sharp surfaces.



Classical potential flow $\Delta\Phi = 0$

The neutron mass current is

$$\boldsymbol{\rho_n} \equiv n_n m_n \boldsymbol{v_n} = \frac{1}{\mathcal{V}_{\text{cell}}} \int_{\text{cell}} n_n(\boldsymbol{r})\nabla\Phi(\boldsymbol{r})$$
$$= n_n^s m_n \boldsymbol{V_n}$$

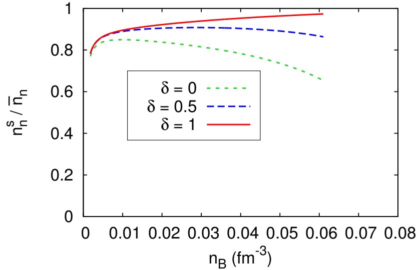
n_n^s is the **superfluid density**
$\boldsymbol{v_n}$ is the true velocity

Martin&Urban,Phys.Rev.C94, 065801(2016)

Classical potential flow past obstacles

Permeability of the clusters:
$\delta = 0$ no superfluidity,
$\delta = 1$ superfluidity everywhere.
Martin&Urban,Phys.Rev.C94,065801(2016)

Added perturbations from different clusters are negligible.
Epstein, ApJ333, 880 (1988)



The potential flow past a single cluster can be solved analytically:

$$\frac{n_n^s}{n_n} = 1 + 3\frac{\mathcal{V}_{\rm cl}}{\mathcal{V}_{\rm cell}}\frac{\delta-\gamma}{\delta+2\gamma} \Rightarrow 1 - \frac{3}{2}\frac{\mathcal{V}_{\rm cl}}{\mathcal{V}_{\rm cell}} \leq \frac{n_n^s}{n_n} \leq 1 + 3\frac{\mathcal{V}_{\rm cl}}{\mathcal{V}_{\rm cell}}$$

Magierski&Bulgac,Act.Phys.Pol.B35,1203(2004); Magierski, IJMPE13, 371(2004)
Sedrakian, Astrophys.Spa.Sci.236, 267(1996); Epstein, ApJ333, 880 (1988)

The superflow is found to be only **weakly perturbed** by clusters.
However, the strong coupling regime is usually not reached.

Suppression of band structure effects by pairing?

By solving the HFB equations with Bloch boundary conditions at $\bar{n} = 0.03$ fm^{-3}, Watanabe&Pethick found that the superflow

- is more strongly perturbed than predicted by classical hydrodynamics,
- **but much less than band calculations without pairing**: $n_n^s/n_n \sim 60 - 70\%$ instead of $\sim 7\%$.

Watanabe&Pethick,PRL119,062701(2017)

But questionable approximations were made:

- 3D body-centered cubic lattice replaced by a **1D lattice**
- **no effective mass** $m_n^\oplus(\boldsymbol{r}) = m_n$
- **Fourier components of** $U(\boldsymbol{r})$ **treated independently**
- pairing potential $\Delta(\boldsymbol{r})$ not solved self-consistently but fixed
- numerical extraction of n_n^s from second derivatives of the energy

Role of pairing further examined

- **full 3D** calculations (body-centered cubic lattice)
- same model as in 2012 with $m_n^{\oplus}(\boldsymbol{r})$ and $U(\boldsymbol{r})$
- pairing included in the **BCS approximation**
- **analytical** extraction of n_n^s

Δ (MeV)	n_n^s/n_n (%)
1.59	7.50
1.11	7.50
0.770	7.51
0.535	7.54
0.372	7.60
0.259	7.66
0.125	7.71
0.0869	7.80

$\bar{n} = 0.03$ fm^{-3}

1550 neutrons in the W-S cell

lattice spacing 47.3 fm

3D grid: $25 \times 25 \times 25$ ($\delta r \sim 0.95$ fm)

1265 bands $\times$ 1360 $\boldsymbol{k}$

Including pairing is computationally costly, but results are essentially the same as those obtained without.

Superfluid reservoir and giant pulsar glitches

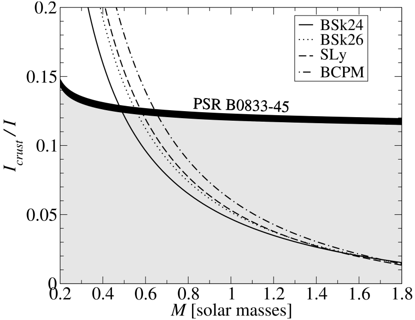
The depletion of the superfluid reservoir in the crust leads to a very **stringent pulsar glitch constraint**.

Chamel&Carter,MNRAS368,796(2006)

The inferred mass of Vela is much lower than expected from supernova simulations and known neutron-star masses.

At such central densities ($\bar{n} \approx 0.23 - 0.33$ fm^{-3}), the equation of state is fairly well constrained by laboratory experiments.

Delsate et al., Phys.Rev.D94, 023008(2016)



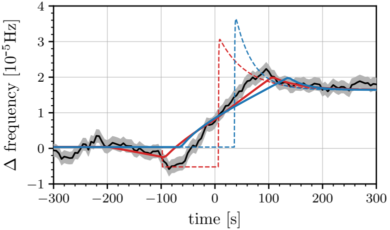
The superfluid in the crust does not carry enough angular momentum. Some superfluid in the core must be also involved.

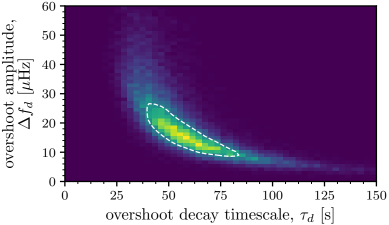
Andersson et al., PRL 109, 241103; Chamel, PRL 110, 011101 (2013)

Glitch rise

Timing of the Crab and Vela pulsars have recently revealed very peculiar evolutions of their spin frequency during the rise of a glitch.

- Analyses of a Vela glitch in 2016 suggest a rotational-frequency **overshoot and a fast relaxation ($\sim$ min)** following the glitch.





Ashton, Lasky, Graber, Palfreyman, Nature Astronomy 3, 1143 (2019)

Glitch rise

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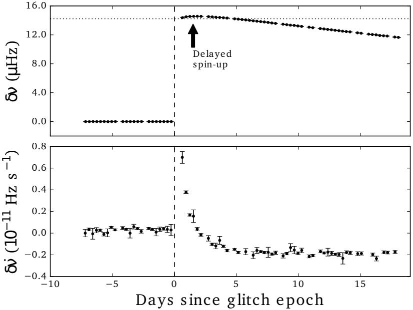
  Ashton, Lasky, Graber, Palfreyman, Nature Astronomy 3, 1143 (2019)

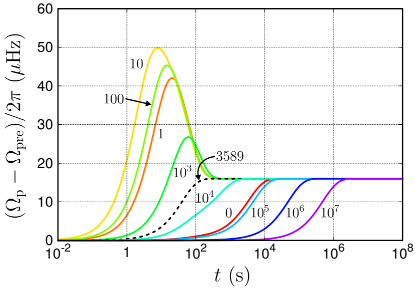
- A **delayed spin-up** has been detected in the 1989, 1996 and 2017 Crab glitches.



Shaw et al., MNRAS, 478, 3832 (2018)

Role of vortex pinning to fluxoids

These differences can be interpreted from the interactions between superfluid vortices and proton fluxoids in neutron-star cores.



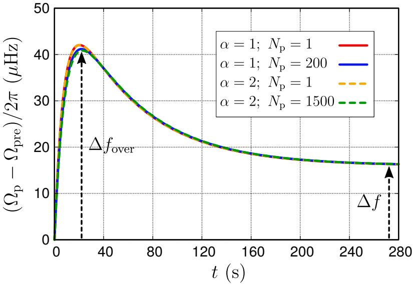
The number N_p of fluxoids attached to vortices turns out to be a key parameter governing the global dynamics of the star:

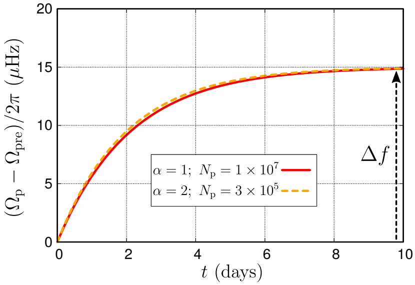
- $N_p < N_p^{crit}$: overshoot $\Delta\Omega_{over} < \Delta\Omega/(1 - I_n^{free}/I)$,
- $N_p < N_p^{crit}$: smooth spin-up on a longer timescale.

Sourie&Chamel, MNRAS 493, L98 (2020)

Role of vortex pinning to fluxoids

The behavior of Vela and Crab glitches can be reproduced:





However, this neutron-star model remains very simplified:

- Newtonian approach
- physical reason for different N_p remains to be investigated
- $N_\mathrm{p}^\mathrm{crit}$ depends on poorly-known mutual friction.

Alternative explanations:

Haskell et al., MNRAS 481, L146 (2018)

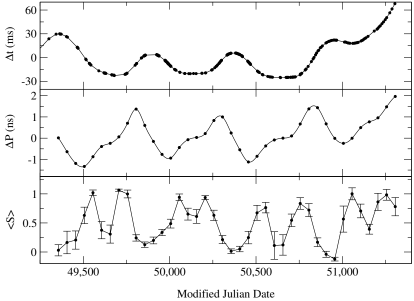
Gügercinoğlu&Alpar, MNRAS 488, 2275 (2019)

Neutron star precession

Long-term cyclical variations of order months to years have been reported in a few neutron stars: Her X-1 (accreting neutron star), the Crab pulsar, PSR 1828−11, PSR B1642−03, PSR B0959−54 and RX J0720.4−3125.

Example: Time of arrival residuals, period residuals, and shape parameter for PSR 1828−11

Stairs et al., Nature 406(2000),484.



These variations have been interpreted as the signature of **neutron star precession**.

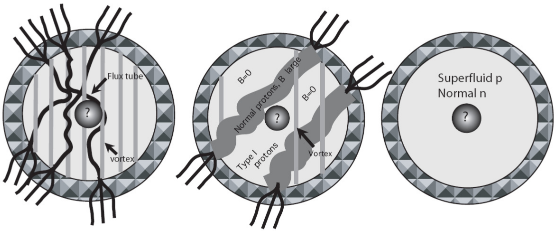
Precession and superfluidity

For a non-superfluid star with deformation $\epsilon = \Delta I / I$,

$$P_{\text{prec}} = \frac{P}{\epsilon} \gg P$$

For a superfluid star with pinned vortices

$$P_{\text{prec}} = \frac{I_{\text{pin}}}{I} P \ll P$$



Link, Astrophys. Space Sci.308,435 (2007)

Observations of precession could shed light on superfluidity. On the other hand, precession may trigger instabilities that could unpin vortices.

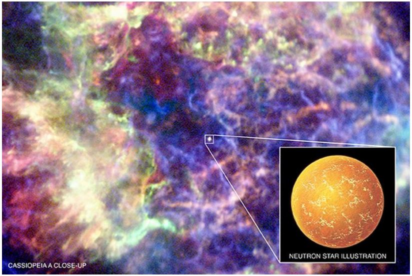
Glampedakis,Andersson,Jones,PRL100,081101(2008)

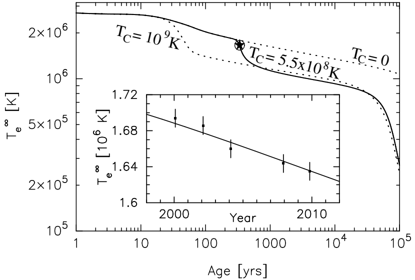
Neutron-star cooling

Other observations support the existence of neutron-star superfluids:

- Observations of Cassiopeia A provide evidence for $^3PF_2$ neutron superfluidity in neutron-star cores.

  Page et al., PRL 106, 081101; Shternin et al.,MNRAS 412, L108





- Observations of quasi-persistent soft X-ray transients provide evidence for $^1S_0$ neutron superfluidity in neutron-star crusts.

  Shternin et al., Mon. Not. R. Astron. Soc.382(2007), L43

  Brown and Cumming, ApJ698 (2009), 1020

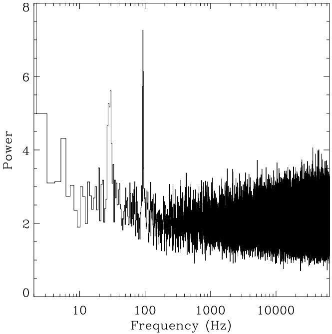
Asteroseismology of neutron stars

The presence of superfluids and superconductors in neutron stars leads to the existence of **new oscillations modes**.

Quasiperiodic oscillations (QPOs) have been detected in the X-ray flux of giant flares from soft gamma-ray repeaters (strongly magnetized neutron stars)

Example: SGR 1806−20

Strohmayer&Watts, ApJ653,593 (2006)

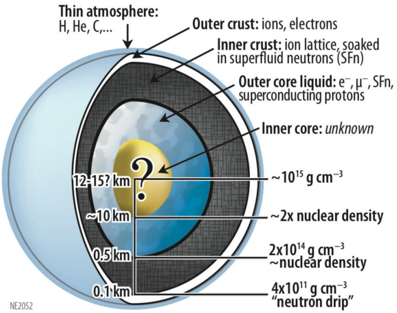


These QPOs are thought to be the signatures of **superfluid magneto-elastic oscillations** but detailed models are still lacking.

Gabler al., PRL 111, 211102 (2013)

Summary

Nuclear superfluidity and superconductivity in neutron stars were predicted as early as 1959.



Neutron-star superfluidity and superconductivity are supported by independent observations (pulsar glitches, cooling).

However, many aspects still remain poorly understood.

Prospects: gravitational-wave asteroseismology (3d generation).

The main challenge is to relate the local nonrelativistic dynamics of vortices and fluxoids at the nuclear scale ($\sim$ 10 fm $=$ 10^{-14} m) to the global general-relativistic dynamics of the star ($\sim$ 10 km).